The role of analogies and anchors in addressing students' misconceptions with algebraic equations

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This pilot study explores the effectiveness of a strategy for overcoming post-primary students' misconceptions within the topic of algebra. Although central to the study of mathematics, algebra can be an area of difficulty for many students. A misconception is typically classified as flawed understanding of a concept causing repeated errors, and the prevalence of algebraic misconceptions has been well documented in the literature. Previous studies in probability and statistics have shown that the use of analogies to misconception-prone situations can act as anchors to effect conceptual change and assist in correcting misconceptions. This project follows a framework that has been successfully utilised in previous research, with two versions of a test and interviews conducted with 25 students. The results identified the presence of, and links between, some common misconceptions among the students regarding their knowledge of algebraic equations and also provided insight regarding an effective technique for addressing these misconceptions.

Introduction

An individual's reading comprehension is defined as their ability to process text, understand its meaning, and to integrate it with what they already know (Grabe, 2009). The relationship between reading comprehension and mathematics has been previously explored in the literature (e.g. Pimperton & Nation, 2010; Harlaar, Kovas, Dale, Petrill & Plomin, 2012). Pimperton and Nation (2010) provided evidence that although students with poor reading comprehension skills did not differ significantly from students in a control group on a numerical operations task, they did perform at a significantly lower level on a mathematical reasoning task. A mathematical reasoning task is one that requires students to manipulate and analyse representations, diagrams, symbols or statements, to draw conclusions based on evidence or assumptions (Batista et al., 2017). To analyse statements within mathematics, students must demonstrate proficiency in the language of mathematics, which utilises a combination of everyday words, mathematical words, and symbolism to communicate meaning. According to Harlaar et al. (2012, p.624), "the importance of language for mathematics is particularly relevant in the middle school years", as at this stage of their mathematics education students are tasked with developing mathematical reasoning skills, mastering complex mathematical procedures and comprehending more abstract mathematics.

A key area of mathematical study where students are expected to develop mathematical reasoning is in the study of algebra which, according to various authors (e.g. Christianson et al., 2012), is inundated with numerous misconceptions that only add to the difficulties that students encounter when studying it. Therefore, this paper reports on the effectiveness of a strategy for overcoming post-primary students' misconceptions within the topic of algebra, which may assist students when attempting to develop their mathematical reasoning skills and their mastery of the language of mathematics.

Algebra is the use of letters and symbols to represent unknown numbers and to illustrate mathematical relationships between quantities in formulae and equations. Following the introduction of a new mathematics curriculum in Ireland in 2010, the syllabus was partitioned into five general strands – statistics and probability, geometry and trigonometry, number, algebra and functions (NCCA 2015, 2016). Despite its position as a standalone strand in the syllabi, algebra has applications in almost all areas of mathematics as it facilitates the computation of unknown values through the use of symbolic representation. Thus, algebra has a pivotal role to play in mathematics and success in mathematics hinges on a sound understanding of algebraic concepts. Fundamental to algebra are the concepts of variables (or letters standing for numbers) and equivalence (Knuth et al., 2005; Weinberg et al., 2016), and mastery of these concepts are essential for progression within algebra as well as general problem solving performance. Unfortunately, algebra is often viewed as an area of difficulty among students (Kieran, 1992; Knuth et al., 2005; McNeil et al., 2010; Stephens, 2006; Welder, 2012) with the unfortunate consequence that it hinders their ability to progress in mathematics, as well as in other school subjects or employment opportunities (Ladson-Billings, 1998).

The cause of mathematical errors, in general, has long been a subject of much research and is a continuing concern in mathematics. A misconception can be described as "a student conception that produces a systematic pattern of errors" (Smith et al., 1994, p.119). Distinction must be made between misconceptions and mistakes, as errors can be caused merely by mistakes, for example due to haste. However, misconceptions indicate flawed understanding, recurrently producing erroneous results (Khazanov, 2008). Across different areas of mathematics, research has uncovered many misconceptions that affect students' performance. Misconceptions are particularly prevalent in algebra with the interpretation and use of literal and equality symbols (Knuth et al., 2005; Welder, 2012), which can produce persistent errors in students' treatment of algebraic equations in many contexts, including in the formulation of equations from worded problems (Clement, 1982; Wollman, 1983).

Prior studies of algebraic misconceptions centre around their exposure and prevention, with little emphasis on correction once identified. Yet, misconceptions can be very resilient and difficult to overcome (Clement et al., 1981; Fast, 1997; Welder, 2012) and attempts must be made by teachers and educators to correct as well as to avert them. In probability and statistics, a strategy for dealing with misconceptions, once embedded, was investigated by Fast (1997, 2001) and Lee-Chua (2002, 2003), who used analogies to misconception-prone situations as anchors to reconstruct students' understanding of certain concepts. Their research found this method to be quite effective in overcoming probabilistic and statistical misconceptions. The focus of this study is to investigate if analogous, anchoring situations could assist students in overcoming misconceptions in algebra and, in particular, in the area of algebraic equations, given the significance of algebra in the study of mathematics.

Common misconceptions within algebraic equations

Students' difficulty in translating from worded problems to algebraic equations manifests itself in the common 'reversal error', which has received significant attention in mathematical research (Christianson et al., 2012; Clement, 1982; Wollman, 1983). This is where students transpose the coefficients of the unknown values, resulting in an equation with a different mathematical meaning to the problem at hand. For example, in Clement's (1982) study, when asked to write an equation to represent the condition that "there are six times as many students as professors at this university" using S for the number of students and P for the number of professors, 37% of college engineering students answered the question incorrectly. The typical wrong answer was $6S = P$ rather than the correct $S = 6P$. A similar but more complex formulation question showed even higher incidence of the 'reversal error'.

Clement (1982) suggested that different levels of cognition are at play in students' approaches to the problem. One method he referred to as 'word order matching' is where students match the order of words in a statement directly onto symbols in the equation. Alternatively, he proposed that a 'static comparison approach' is adopted, where the larger number is placed with the larger group to show that it is the larger group. In this case, students are considering the size of one group relative to the other and, although the resulting equation is still incorrect, there is nonetheless deeper reasoning involved than with 'word order matching' (Clement et al., 1981; Clement 1982). Wollman (1983) referred to this approach as 'set match', since 'sets' of students are compared to 'sets' of professors, and both Wollman and Clement found that this error is much more entrenched than 'word order matching' as there appeared to be other misconceptions present around the algebraic symbols themselves – letters and equality sign.

Another well-documented misconception, first uncovered by Küchemann (1981), is that letters in algebra stand for objects or labels rather than numbers. In the students-and-professors 'reversal error', it appears that S is used as a label for 'students' rather than the number of students and P is a label for 'professors' rather than the number of professors (Clement et al., 1981; Clement, 1982; Wollman, 1983). This misconception is common and often arises if the first letter of an object is used to represent some unknown quantity of that object, such as a for 'apple' or b for 'banana', and so the letter is considered an abbreviation for the word (MacGregor & Stacey 1997; McNeil et al., 2010; Mooney et al., 2004). McNeil et al. (2010) showed that students performed better when letter symbols used to represent unknown quantities differed completely from the names of the items themselves. They also conjectured that the misconception could be elicited through inaccurate explanations by teachers and other sources. Küchemann (1981) pointed out that even bearing this misconception, students will obtain the correct answer to the question $2a + 5b + a$ (3 apples and 5 bananas). However, in the case where c stands for the 'cost of cakes' and b stands for the 'cost of buns', students misinterpret the meaning of the expression $4c + 3b$ as being '4 cakes and 3 buns' (Küchemann, 1981).

Additionally, with reversals like $6S = P$ resulting from a 'static comparison approach', the equals sign is used incorrectly, because although students understand that the group of students is larger than the group of professors, the equals sign is used to denote a relationship or ratio rather than an equality (Clement et al., 1981; Clement, 1982; Wollman 1983). Misconceptions around the equals sign are common, where students view the sign as 'operational' rather than 'relational', and so understand it as the announcement of the answer rather than a symbol of equality and balance (Knuth et al., 2005, 2006; Li et al., 2008; Stephens, 2006).

Attempts to overcome misconceptions around algebraic equations

Many attempts have been made to address the previously mentioned misconceptions. The success of these varied, with much of the advice focusing on the detection and avoidance of the misconceptions in the first instance. In terms of translation from words to equations, it is noted that students are seldom asked to formulate an equation but are normally given the equation and time is dedicated to the steps involved in solving it (Clement et al., 1981; Clement, 1982; Wollman, 1983). To help develop the necessary skills to effectively convert from worded to algebraic form, it is important to provide students with plenty of opportunities to practise these types of problems, carefully choosing examples that require an understanding of the number on which to operate to produce equality, and highlighting these aspects of the process (Clement et al., 1981). Christianson et al. (2012), in fact, researched the use of practice and repetition and found that practice in completing many of the students-and-professors type problems, without any intervention, appeared to have a very positive effect on performance. This would suggest that the 'reversal error' is not as deep-rooted as previous studies, which only used a very small number of questions, indicated (Christianson et al., 2012). Christianson et al. (2012) also proposed that students, after they have formulated the equation, should substitute in numbers for the letters to ensure that their equation makes sense mathematically.

Teaching approaches are also very important in addressing misconceptions. It is imperative that curricula, textbooks and teaching materials are thoroughly examined to reduce the possibilities of misconceptions developing (McNeil et al., 2010; Welder, 2012). In MacGregor and Stacey's (1997) study, interestingly the students from one of the three schools had a very high incidence of interpreting letters as objects and upon further investigation, it transpired that in the textbooks used at this school the concept that "*c* could stand for 'cat', so 5*c* could mean 'five cats'" was used. It was noteworthy that students from the other schools in the study did not hold this misconception, with only two cases of this appearing in their answers (MacGregor & Stacey, 1997).

Using anchors to overcome misconceptions

> The logical approach to overcoming a misconception is to generate or find a situation to which the student will respond correctly, so that an anchor of mathematically correct knowledge is established on which the reconstruction process can begin (Fast 2001, p.196).

As mentioned, misconceptions are prevalent across different areas of mathematics, and a method employed in the area of probability and statistics to target misconceptions and reconstruct students' understanding of particular concepts was the use of analogous, 'conceptually isomorphic' situations that serve as anchors (Fast, 1997, 1999, 2001; Lee-Chua, 2002, 2003). This approach replicated a model successfully trialled by Clement in the field of physics (1987b, cited in Fast, 1997). Fast's research involved creating an instrument with two tests for completion by high school students and university student teachers. The first test contained questions on misconception-prone probability situations. The second test contained corresponding analogous questions, which were carefully designed with the aim of correcting students' knowledge if the misconceptions existed (Fast, 1997, 1999, 2001). In creating the second test, he employed techniques such as looking at the problem from a different angle, removing misconception triggers, using more concrete or recognisable situations, or using different numerical values to demonstrate extreme cases.

When addressing probability misconceptions among university students, Cox and Mouw (1992) established anchors by adding or removing 'cues' or situations to target the misconceptions, with the intention of unsettling students' previously held ideas, thereby forcing them to confront and reconsider their reasoning. Probabilistic and statistical misconceptions were the areas of attention for Lee-Chua (2002, 2003), who used similar methods to Fast in designing her anchoring questions for university students and, in some cases, providing less information on a corresponding second test question served as an anchor to elicit correct understanding. Alternative anchoring situations included making the correct response more discernible, or the incorrect response appear absurd, in order to guide students to the correct answer and conceptual understanding (Lee-Chua 2002, 2003).

With the exception of Cox and Mouw's (1992) study, overall the results of the research show that the use of anchors in targeting and assisting students in overcoming misconceptions in probability and statistics is quite effective. These results are very significant, given the complexity of reconstructing understanding and the resilience of probabilistic and statistical misconceptions (Fast, 1997, 1999, 2001; Lee-Chua, 2002, 2003).

Method

Research aims and questions

The aim of this research is to investigate the effectiveness of using analogies to act as anchors in helping students to overcome misconceptions pertaining to algebraic equations. The following research questions will help to guide this study.

- Which misconceptions do students hold in relation to algebraic equations?
- Can anchoring situations, analogous to misconception-prone situations, correct those misconceptions?

Specifically, the study focuses on the following three misconceptions:

i. the 'reversal error' in translating from words to algebraic equations;
ii. the concept of equivalence;
iii. the misunderstanding of algebraic 'letters as objects'.

Research design

The approach used in this study followed the framework advocated by Fast (1997, 1999, 2001) and Lee-Chua (2002, 2003) in the field of probability and statistics, following on from the work of Clement (1987b cited in Fast, 1997) on misconceptions in physics. Both Fast and Lee-Chua generated instruments with two tests, the first containing questions on misconception-prone situations, the second containing corresponding, analogous questions designed with the aim of correcting those probabilistic and statistical misconceptions. Follow up interviews were carried out to determine if the anchors had been successful.

Ethical approval was received from the ethics committee at the University of Limerick. Information sheets were issued to the principal of the school in which the research would be conducted and then to students and their parents/guardians. A pilot study was carried out and consent forms were received from students and their parents/guardians to allow them to participate in both the written tests and interviews.

The participants

The research was conducted with a second-year class of post-primary school students in the Republic of Ireland, aged 13-14 years. 25 participants - 8 boys and 17 girls - completed the tests and follow-up interviews were conducted with eight of those participants. These students had all studied algebra in their first year of post-primary school and should have seen the concept of a variable in primary school. The participants were selected by convenience sampling, whereby a full available second year class participated in the study. As the first author was not the teacher of a second-year class at the time of this study, the second-year class of another teacher participated in the research. The tests were piloted in advance with six 16 year old students.

The instrument

The instrument consisted of two tests - *Version A* and *Version B* – and follow up interviews with a selection of participants who completed the tests. A mixed methods approach was adopted, whereby both quantitative and qualitative data was collected. To collect the quantitative data, participants completed the *Version A* test, before completing *Version B*. To collect the qualitative data, the first author conducted follow up interviews with a selection of those participants.

Version A contained eight questions, which targeted specific misconception-prone situations relating to algebraic equations, as identified in previous studies. Questions 1, 2 and 3 on *Version A* were adapted from Clement's (1982) research in converting from worded to symbolic representation, commonly resulting in the 'reversal error'. Questions 4, 5 and 6 related to misconceptions around the equals sign. Question 4 was adapted from examples discussed by Li (2008) and Questions 5 and 6 were modified from research by Stephens (2006). Finally, Questions 7 and 8 were concerned with the algebraic misunderstanding of 'letters as objects' and were adaptations of Küchemann's (1981) examples.

All questions were multiple-choice style, with the exception of Question 7. Question 7 required participants to explain the meaning of a given algebraic expression to determine if they held the 'letters as objects' misconception (see Appendix A). The other 7 questions were multiple-choice style with three possible answers, only one of which was correct, and participants were requested to choose the correct answer. One of the incorrect answers targeted the misconception insofar as it would be the typical answer given by a student holding that misconception, and the second incorrect answer served no purpose but to have a third option. Participants were subsequently asked to explain why they chose the answer they did, and the questions were followed by a confidence scale, where participants indicated how confident they were that they had completed the question successfully, choosing from 'just a guess', 'not very confident', 'fairly confident' to 'I'm sure I'm right'.

Version B also contained eight questions, which were all multiple-choice style with three possible answers, one of which was correct and two incorrect. Again, one of the incorrect answers would be the expected answer given by a student with the targeted misconception and the function of the second incorrect answer was merely to provide a third choice. As in *Version A*, it was required that participants provide an explanation for the chosen answer, followed by the same confidence scale. Each question was similar or analogous to its counterpart on *Version A* but was designed to 'anchor' participants if they held the specific misconception, with the hope of bringing them to the correct understanding. The methods employed in designing the *Version B* questions were to use different numerical values, to rearrange the order of terms, to include subtle prompts, to use an equation rather than an expression or to remove one variable to make the correct answer more recognisable.

Figure 1 shows Question 1 from *Version A* and its analogous counterpart from *Version B*. Smaller numbers have been used on *Version B*, in addition to the prompt to consider the number of girls in terms of *b*, the number of boys, *before* letting *g* represent the number of girls. For further examples of *Version A* and *Version B* questions, see Appendix A: Tests, which presents the full set of questions.

The procedure

A pilot test was carried out in advance with students, who confirmed their understanding of the questions and had no suggestions in relation to wording or improvement of the tests. The students felt that the questions were clear and did not require further clarification. Following this, the tests were distributed by the teacher of the participant class during a scheduled mathematics class. Completion of the *Version A* test took approximately 15 minutes. The test was then collected and participants completed *Version B*, which also took approximately 15 minutes.

Question 1: (Version A)

1. "There are 20 times as many students as teachers in the school". Using s as the number of students and t as the number of teachers, which of the following equations is correct.

    a. $20s = t$
    b. $s = 20t$
    c. $4s = 5t$

    Why?____

Question 1: (Version B)

1. "There are twice as many girls as boys in a class". If b is the number of boys, consider the number of girls in terms of b. Now let g be the number of girls in the class. Which of the following equations is correct?

    a. $g = 2b$
    b. $b = 2g$
    c. $b = g$

    Why?____

Figure 1: Example of question from both *Version A* and *Version B* tests.

Following completion of the tests, the data was analysed. Each question on *Version A* and its corresponding counterpart on *Version B* was recorded as 'correct', 'incorrect' or 'not answered'. This information was collated to determine numbers of participants who answered correctly/incorrectly or not at all on each question and to show where possible misconceptions might be present. Participants who answered incorrectly to a question on *Version A* but correctly to the analogous question on *Version B* were of interest, as this showed potential effectiveness of the anchors. The tests were studied using the justification of answers given to ascertain if possible misconceptions and anchors were true misconceptions and anchors (defined in the 'Results' section which follows). The confidence scale was used to provide an overall picture of participants' confidence levels on the *Version A* test in comparison to the *Version B* test. It was also used to distinguish 'true' anchors from possible anchors. Eight participants were selected for follow-up interviews, each of which lasted less than 5 minutes, to obtain further information on the effectiveness of the anchors to dispel the misconceptions. Most participants selected had answered a question on *Version A* incorrectly, indicating a possible misconception, but had answered the analogous question on *Version B* correctly. The interviews were carried out and recorded to determine if the anchoring situations had been successful.

Results

Responses on Version A and Version B

There were 200 potential responses on each test (25 participants x 8 questions). On *Version A*, 17 questions received no response. 12 of these non-responses were on the last two questions on the test, indicating that the time allotted may not have been sufficient for some participants to complete all questions. Since it is impossible to determine if a non-response signifies an inability to answer the question and potentially the presence of a misconception these questions have to be ignored and cannot be included as *possible* misconceptions. Of the answers on *Version A*, 74 were correct and 17 were blank, leaving 109 as possible misconceptions. There was no response on just one question on *Version B* and of the 199 responses provided, 100 were correct. Hence, 50% of responses given on the analogous *Version B* questions were correct (100/200) in comparison to 37% of correct responses on the target *Version A* questions (74/200).

For analysis, a correct response was recorded as 1, an incorrect response as 0 and no response as x (see Table 1). 20 of the 25 participants had more correct responses on *Version B* than on *Version A*. 2 participants had the same number of correct responses on *Version A* and *Version B*. For 3 participants, the number of correct responses on *Version A* exceeded the number of correct responses on *Version B* by one answer.

The breakdown of correct responses per question is shown in Table 1 and Figure 2. In Questions 3 and 4, there were more correct responses on *Version A* than on *Version B*. In both cases, this was just by one answer. All other questions had more correct responses on *Version B*, as would be expected. It can be seen from Figure 1 that some questions appeared to be more effective than others in eliciting a correct response on *Version B*, comparative to the corresponding counterpart on *Version A*. Of particular note is Question 7 relating to the 'letters as objects' misconception. Here, the largest disparity between correct answers on corresponding questions can be seen, with 22 correct responses on the *Version B* question in comparison to 6 correct on the equivalent *Version A* question.

Table 1: Data from tests *Version A* and *Version B*

Question	1		2		3		4		5		6		7		8		Total correct	
Version	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B		
Participant																	A	B
1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	x	1	6	7
2	0	1	0	0	1	1	1	1	0	1	1	1	0	1	0	0	3	6
3	x	0	0	0	x	0	1	1	1	1	1	1	x	1	x	0	3	4
4	0	0	0	0	0	0	1	1	1	1	x	x	0	0	1	0	3	2
5	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	4	4
6	0	1	0	0	0	0	1	1	1	1	1	1	0	1	0	0	3	5
7	0	0	0	0	0	0	1	1	1	1	1	1	0	1	0	0	3	4
8	1	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	2	2

9	0	0	0	1	0	0	1	1	0	1	1	1	1	1	x	0	3	5
10	0	1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	4	5
11	0	0	0	0	0	0	1	1	1	1	1	1	0	1	0	1	3	5
12	0	0	0	0	0	0	1	1	1	1	1	1	0	1	0	0	3	4
13	0	0	0	0	0	0	1	1	0	0	0	1	0	1	0	0	1	3
14	0	0	0	0	0	0	1	1	0	0	1	1	x	1	x	0	2	3
15	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	2	1
16	0	0	0	0	0	0	1	0	0	1	1	1	0	1	x	1	2	4
17	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	5	4
18	0	0	0	0	0	0	1	1	0	1	1	1	0	1	1	1	3	5
19	1	1	0	0	1	0	1	1	0	1	1	1	x	1	0	0	4	5
20	0	0	0	0	0	0	1	1	1	1	1	1	0	1	0	0	3	4
21	0	0	0	0	0	0	1	1	0	1	1	1	1	1	0	0	3	4
22	0	0	0	0	0	0	1	1	0	0	1	1	0	1	x	0	2	3
23	0	0	0	0	0	0	1	1	0	1	1	1	0	1	x	0	2	4
24	0	0	0	0	0	0	1	1	0	0	1	1	0	1	0	0	2	3
25	0	0	0	0	0	0	1	1	1	1	1	1	x	1	x	0	3	4
Total correct	3	5	1	2	3	2	25	24	12	18	21	23	6	22	3	4	74	100
Possible mis-conceptions*	21		24		21		0		13		3		15		14		111	
True mis-conceptions	21		24		21		0		0		2		12		14		94	
Possible anchors		3		1		0		0		7		2		12		1		26

* The terms 'Possible misconceptions', 'True misconceptions' and 'Possible anchors' are explained in the paragraphs above and below.

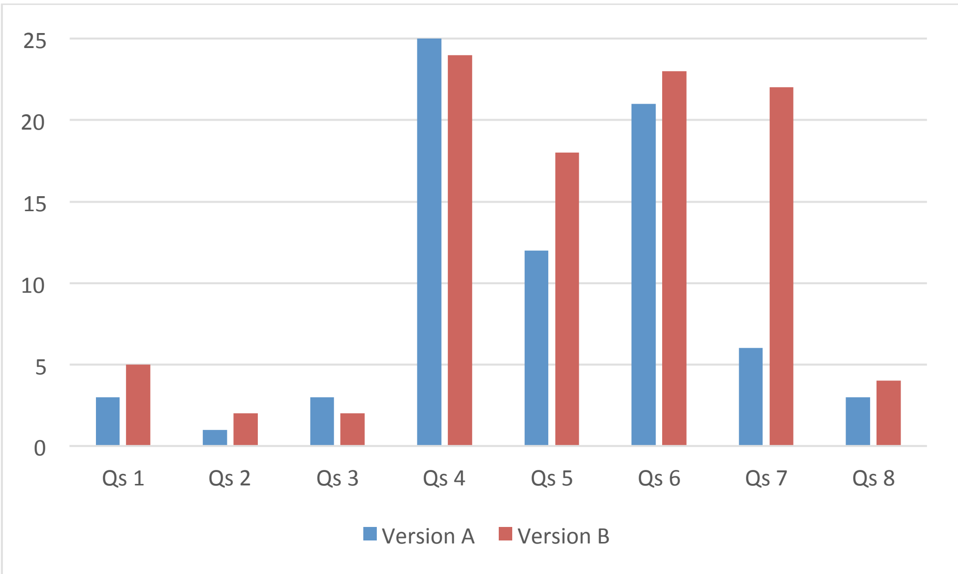

Figure 2: Number of correct responses per question on *Version A* and *Version B*

Possible misconceptions

Incorrect responses on *Version A* were considered possible misconceptions, 'possible' as a mistake can occur due to haste or an oversight and does not necessarily indicate a misconception (Khazanov, 2008). The results of *Version A* indicated 111 possible misconceptions. To classify these, recall that the aim of Questions 1, 2 and 3 was to unearth the 'reversal error', Questions 4, 5 and 6 targeted 'equivalence' and Questions 7 and 8 were focused on the 'letters as objects' misconception. Table 2 shows a breakdown of incorrect responses relating to these misconceptions and the percentage of questions answered incorrectly. Based on these figures, the 'reversal error' appears to be very prevalent, as does the misinterpretation of letters as objects or labels. The figures appear to suggest the existence of some possible misconceptions around the equals sign, albeit to a much lesser extent than the other misconceptions.

Table 2: Number of incorrect responses/possible misconceptions on *Version A*

	Incorrect responses		
Misconception	Reversal error	Equivalence	Letters as objects
Related questions	Questions 1, 2, 3	Questions 4, 5, 6	Questions 7, 8
No. of possible misconceptions	66	16	29
Total no. of attempted questions	73	73	38
% of possible misconceptions	90.41%	21.92%	76.32%

Possible anchors

Where a participant answered incorrectly on a target *Version A* question, implying a possible misconception, but responded correctly to the analogous *Version B* question, it was considered a possible anchor for conceptual change, as the participant appeared to have applied different reasoning to answer the questions despite the similar focus of the questions. This may signify a possible shift in thinking, and thus a potential anchoring condition. This happened 26 times, as indicated in the shaded cells in Table 1, which also shows the questions to which those possible anchors relate. As illustrated in Table 1 and Figure 1, Question 7 (see Appendix A) produced the largest number of possible anchors at 12.

True misconceptions

Participants were asked to justify their answers on all questions on both tests. On examining those justifications, a large number of the *possible* misconceptions from incorrect responses on *Version A* did appear to be true misconceptions. Where participants chose the incorrect response expected if the misconception was held but without any justification, the authors assumed this misconception to be held. Of the 111 possible misconceptions suggested, 94 of those appeared to be true misconceptions. It became apparent from examining the justifications provided to the incorrect answers on Question 5 of *Version A* that a misinterpretation of the question rather than a conceptual misunderstanding was the reason for the errors. Overall, the results on Questions 5 and 6 were positive as they indicated an absence of misconceptions around the equals sign when provided with an algebraic equation in one variable. On the other hand, the majority of questions that targeted the 'reversal errors' and 'letters as objects' misconceptions turned out to be true misconceptions (see Table 1).

True anchors

Of the 26 possible anchors, 9 related to Questions 5 and 6, thereby reducing the number of possible anchors to 17. Analysis of the justifications on those 17 *Version B* questions rendered a more accurate picture of the true anchoring effect of the analogous situations. To define a 'true' anchor, it was decided that where participants answered incorrectly on a *Version A* question but correctly on the analogous *Version B* question *and* either displayed correct conceptual understanding or a shift in thinking through their justifications *or* when no justification was given they indicated 'fairly confident' or 'I'm sure I'm right' on the confidence scale, this would be considered a true anchor. Of those 17 possible anchors, 10 appeared to be true anchors based on the definition stated. Overall, this gives a low anchoring rate of 0.106 (true anchors/true misconceptions, (10/94)). However, the authors felt that this was a more accurate method of measuring the effect of this approach than calculating possible anchors/possible misconceptions, as was used in previous studies (Fast 1997, 1999), when it had been identified in the same research study that only a portion of these were true anchors and true misconceptions. In any case, the anchoring Question 7 on *Version B*, which related to the 'letters as objects' misconception, was very effective in producing an anchor of conceptual change with an anchoring effect of 0.666 (8/12).

Confidence levels

Assigning numerical scores to the confidence scale following each question allowed comparison of confidence levels on *Version A* with confidence levels on *Version B*. These were scored as follows: 'just a guess' (0), 'not very confident' (± 1), 'fairly confident' (± 2), to 'I'm sure I'm right' (± 3), where positive values were assigned to questions that were answered correctly and negative values were assigned to questions answered incorrectly, with 'just a guess' having a score of 0. This gave an overall confidence level of -35 on *Version A* and 22 on *Version B*. The substantial difference of 57 between the two tests shows that in addition to *Version B* eliciting more correct responses, participants also felt more confident regarding their answers.

Interviews

Interviews were carried out with 8 participants. Of those participants, 7 had responded incorrectly to a question on *Version A*, indicative of a misconception, but correctly to the analogous question on *Version B*, indicative of an anchor. The other participant had incorrect responses on Question 5 on both tests and the purpose of the interview was to discuss the responses, which appeared to result from misinterpretation of the questions rather than a misconception. The other 7 participants were, in the first instance, directed to the incorrectly answered question from *Version A* and asked if they still agreed with their response. The analogous, correctly answered question on *Version B* was then presented and participants were questioned if they still agreed with the response given here. In some cases, it was required to ask participants to point out similarities between the questions and consider why different reasoning was applied on the *Version A* and the *Version B* questions. In all cases, participants changed their responses on *Version A* to the correct one. To this end, it could be said that all anchoring questions discussed were successful. However, it was apparent that some required more probing than others before the anchor of correct understanding was established. In any case, the anchoring questions on *Version B* proved very useful in the intervention and in facilitating reflection and subsequent change among participants incorrectly answered questions.

The results of the interviews are summarised in Table 3. Prior to commencing the interview the participants were asked to review their test answers and the check mark denotes that the participant still agreed with their original response whereas the cross denotes that the participant did not agree with their original response and wished to change their answer. With Question 6 it emerged that the incorrect choice was simply a mistake, which the participant recognised and corrected spontaneously when asked if they still agreed with their response. It was clear that there was no misconception present. In the same way, Participant 11 changed their response to Question 8 when they reviewed their original answer. It could be conjectured here, however, that the anchoring question on *Version B* had already fulfilled its role of eliciting change in that participant, without the necessity of directing them back to the *Version A* question. In the two cases where a correct response was changed to an incorrect response on the *Version B* question (Participant 9 and 10), participants selected the correct *Version A* and *Version B* answers when the concept was discussed during the interview using the anchoring questions. The remaining respondents all changed their incorrect responses to the *Version A* questions during the interviews.

Table 3: Results of interviews:

Part ID	Question	Version A (incorrect)	Version B (correct)	Version A (changed)	Success
2	Question 1	✓	✓	Yes	Yes
10	Question 1	✓	×	Yes	Yes
9	Question 2	✓	×	Yes	Yes
13	Question 6	×		Yes	Yes
6	Question 7	✓	✓	Yes	Yes
24	Question 7	✓	✓	Yes	Yes
11	Question 8	×		Yes	Yes

✓ Agreed with answer on test; × Disagreed with answer on test

Limitations of the study

Even though this was only a pilot study to assess the effectiveness of the approach, a limitation of the study was the small number of participants in the tests and interviews. With larger numbers, the results would be more conclusive. Also, a limited timescale did not allow for much piloting. A longer timeframe would have allowed for a number of pilot tests, potentially the production of different versions of the tests, retesting and possible revision of the anchoring questions. It would also have allowed for practice or piloting of the interviews.

Discussion

Overall findings

This project set out to ascertain, firstly, which misconceptions students held in relation to algebraic equations and, secondly, if anchoring situations could correct those misconceptions. In response to the first research question, the study found that participants had a marked difficulty in interpreting literal symbols when given in context and in translating from worded to algebraic equation form. This is not surprising since previous research has highlighted the prevalence of misinterpretation of variables as objects (Küchemann, 1981; MacGregor & Stacey, 1997; McNeil et al., 2010) and the struggle that even third level students have in formulating equations from words, frequently resulting in the 'reversal error' (Clement et al., 1981; Clement, 1982; Wollman, 1983). In general, participants did not hold misconceptions around the equals sign when presented with equations in non-contextualised form. This was a positive result, given the degree of misconception around equality. It was encouraging that there was no evidence of an 'operational' understanding of the equals sign, as had been identified in studies by Knuth et al. (2005, 2006) and Li et al. (2008), among others. However, the use (or misuse) of the equals symbol in contextualised problems is a point for discussion.

Misconceptions

Of note was the prevalence of the 'reversal error' in the results. As suggested by Clement (1982), the 'reversal error' results from the application of 'word order matching' or a 'static comparison approach'. While the initial impulse in selecting an answer may have been the former direct left-to-right mapping of words to symbols, the justification requirement likely compelled students to employ the latter, where consideration of the size of one group relative to the other was needed to explain their reasoning. It was clear from their justifications that, in general, participants applied a 'static comparison approach', comparing group size and placing the larger number with the larger group. In other words, participants were cognisant of the fact that there were more students than teachers and so on. This then raises the question of why participants chose the 'reverse' equations, given their awareness of the relationship between group sizes, as the 'reverse' equation has a different mathematical meaning to the situation to be modelled. If students do not understand that $20s = t$ means that the number of students multiplied by 20 is equal to the number of teachers, and implies that there are many more teachers than students, this will largely impact their ability to problem solve, in particular if they are required to formulate an equation first before solving (Jupri & Drijvers, 2016). Developing problem solving skills through translation type problems is a key aim of mathematics education, with Junior and Leaving Certificate syllabi in Ireland stating that students should be able to "analyse information presented verbally and translate it into mathematical form" (NCCA 2015, p.15; NCCA 2016, p.16).

It was evident from student responses, which is in line with Clement's (1982) observations on the students-and-professors problem, that in formulating equations from words, the algebraic letters were regarded as objects or labels rather than numbers. In fact, explicit statements to this effect were seen in such responses as '20s is 20 students'. Furthermore, it was apparent that the equals sign was generally used to denote a ratio rather than an equality of numbers. For example, the meaning of $20s = t$ to many participants was '20 students for every teacher' and thus s and t were shorthand labels for students and teachers and the equals sign meant 'for every', which is also noted by Wollman (1983). In fact, over both tests the words 'for every', 'per', 'is to', 'is equivalent to', 'ratio' or the ratio symbol appeared in 42% of justifications where the 'reversal error' appeared. That participants had little or no difficulty with the questions targeting equivalence, in which the equations were given, supports Clement's (1982) suggestion of students' greater trouble with formulation than manipulation of equations. In short, the results indicate that the 'letters as objects' misconception, although not specifically targeted in the first three questions on the test but rather the last two, plays a large role in the production of the 'reversal error' and, moreover, while students' responses in non-contextualised problems denote no misconception with the equals sign, there appears to be some ambivalence with the symbol use in translation. This highlights the importance of teachers providing opportunities for students to practise formulating as well as solving equations, as emphasised by Clement (1982) and Christianson et al. (2012). The comparison between the 'equivalence' and 'reversal error' questions also shows the need for teachers to stress the uniformity of the equals sign – it represents equality of numbers in *all* contexts.

Having seen the 'letters as objects' misconception, first identified by Küchemann (1981), emerge in the translation problems through the 'reversal error', attention was then turned to the questions that had solely targeted the 'letters as objects' misconception. The results on these questions reinforced the prevalence of this misconception. On the *Version A* test many participants explicitly described '$3s + 6r$' as '3 sandwiches and 6 rolls' and '5t = 5 tries' and '2c = 2 conversions'. It is essential that teachers are aware of these pitfalls and, from the outset, accustom students to substituting sample numbers into equations to reinforce the real meaning of variables, which are the core concepts of algebra. Furthermore, teachers must be mindful of teaching materials and explanations to ensure that letters are never presented as objects. MacGregor and Stacey (1997) pointed out that in spite of the impulse to use concrete objects to teach abstract ideas, if they are not correctly applied they could serve to engender or reinforce this misconception.

Anchors

The second research question sought to determine if anchoring situations could assist students in overcoming the algebraic misconceptions unearthed by the first research question. It was apparent from the results that the anchors targeting the 'reversal error' were not largely effective in their own right. On the other hand, the anchoring approach was very effective in assisting participants to overcome the 'letters as objects' misconception. It was also shown, however, that the two misconceptions were very much intertwined and that perhaps the 'letters as objects' misconception needs to be corrected first before attempting to address the 'reversal error' misconception. It is also important to note that while the overall anchoring effect on the tests appeared low it was clear from the interviews that, with intervention, the analogous, anchoring questions proved very useful in facilitating conceptual change in participants. Thus, the use of analogies could be a very effective method of dealing with potential misconceptions.

The most noteworthy finding related to Question 7, which targeted the 'letters as objects' misconception. The analogous *Version B* question was a very successful anchor with 6 students responding correctly on *Version A* in comparison to 22 on *Version B*. The target question required participants to explain the meaning of '$3s + 6r$', given that 'sandwiches cost s euro and rolls cost r euros', which would be correctly interpreted as the numerical 'cost of 3 sandwiches and 6 rolls'. It is likely that the misunderstanding of '$3s + 6r$' as '3 sandwiches and 6 rolls' lost its validity when the equation $3s + 6r = 39$ was introduced in the anchoring question. It compelled students to consider the true meaning of s and r since '3 sandwiches and 6 rolls is 39' makes no logical sense. The justifications on *Version B* included references to 'amount of money', 'cost', 'euros', 'value' and 'price', indicating a clear understanding of the variables in the equation. A similar approach could be adopted in designing any anchoring question to correct the 'letters as objects' misconception. This is a simple, but very effective method that could bring students to the correct understanding of variables in algebra.

Interestingly, despite choosing the correct option on *Version B*, one participant wrote 'I'm not sure if the letters stand for euros or sandwiches and rolls'. This was significant as the student had described '$3s + 6r$' as '3 sandwiches + 6 rolls' in *Version A*. Although the participant was unsure whether the letters stood for objects or numbers when confronted with the anchoring question, it showed a clear shift in thinking, a 'perturbance' or 'disruption', similar to that outlined by Cox and Mouw's (1992) research on anchoring probabilistic misconceptions by showing inconsistencies with the logic applied. This was evidence that the anchoring question was having the desired effect of at least 'disrupting' the participant's understanding and forcing them to reconsider their reasoning, even if they were not fully there yet.

Conclusion

A fundamental idea in successful formulation of equations is the correct conceptual understanding of algebraic letters and the equality symbol, and the importance of teaching approaches in this regard have been highlighted within this study. Useful insights could be gained if this research project were to be viewed as a pilot project. A significant finding was that assigning a total numerical value to an algebraic expression served as a very effective anchor for correcting the 'letters as objects' misconception and, from that insight, the possible use of simple manipulation and questioning, as suggested, could have positive implications for the teaching of algebra. Questioning techniques have a key role to play in effective teaching and, while the overall anchoring effect of the tests in this research was low, it emerged from the interviews that the use of analogous, anchoring questions could be a very successful strategy for teachers to use in different contexts to assist students in overcoming misconceptions relating to algebraic equations and thus develop their mathematical reasoning skills. Considering the crucial role that algebra plays in all areas and at all levels of mathematics education and the necessity of a deep understanding of algebra for correct application and authentic development of problem-solving skills, it is essential that different approaches and strategies are explored to correct algebraic misconceptions.

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Appendix A: Tests

Version A

Instructions:
For each question, choose the correct answer (a), (b) or (c).
Then indicate how confident you are about that answer.

1. "There are 20 times as many students as teachers in the school". Using s as the number of students and t as the number of teachers, which of the following equations is correct.

    a. $20s = t$
    b. $s = 20t$
    c. $4s = 5t$

    Why?____
    ____
    ____

    Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

2. "At Emma's restaurant, for every four people who order chicken there are five people who order steak". If c is the number of chicken dinners ordered and s is the number of steak dinners ordered, which of the following equations is correct.

    a. $5c = 4s$
    b. $c = s$
    c. $4c = 5s$

    Why?____

    Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

3. "There are 7 times as many people in Dublin as there are in Limerick". If d represents the number of people in Dublin and l represents the number of people in Limerick, which of the following equations is correct.

    a. $14d = 2l$
    b. $7d = l$
    c. $d = 7l$

    Why?____

    Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

4. If $6 + 9 = x + 4$, which of the following is true:

    a. $x = 15$
    b. $x = 11$
    c. $x = 5$

    Why?____

    Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

5. Given that $2y + 15 = 31$, is the following equation true:
   $2y + 15 - 9 = 31 - 9$

   a. Yes
   b. No
   c. It depends on the value of y

   Why?____

   Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

6. The solution to the equation $3a + 18 = 24$ is $a = 2$.
   What is the solution to $3a + 18 - 6 = 24 - 6$?

   a. $a = 2$
   b. $a = 12$
   c. $a = 18$

   Why?____

   Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

7. "Sandwiches cost s euros and rolls cost r euros in a shop." Suppose I have €50 and I buy 3 sandwiches and 6 rolls, what does $3s + 6r$ stand for?

   Why?____

   Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

8. "In a game of rugby, a team is awarded 5 points for a try and 2 points for a conversion." If t is the number of tries and c is the number of conversions scored by a team in a match, what does $5t + 2c$ stand for?

    a. The total number of tries and conversions scored
    b. 5 tries and 2 conversions
    c. The total score of the team in the match

    Why?____

    Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

Version B

Instructions:
For each question, choose the correct answer (a), (b) or (c).
Then indicate how confident you are about that answer.

1. "There are twice as many girls as boys in a class". If b is the number of boys, consider the number of girls in terms of b. Now let g be the number of girls in the class. Which of the following equations is correct.

    a. $g = 2b$
    b. $b = 2g$
    c. $b = g$

    Why?____

    Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

2. "At Emma's restaurant, for every person who orders chicken there are five people who order steak". If c is the number of chicken dinners and s is the number of steak dinners ordered, which of the following equations is correct.

    a. $5c = s$
    b. $c = s$
    c. $c = 5s$

Why?____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

3. "There are twice as many people in Tipperary as there are in Laois". If t represents the number of people in Tipperary and l represents the number of people in Laois, which of the following equations is correct.

   a. $1t = 1l$
   b. $2t = 1l$
   c. $1t = 2l$

   Why?____

   Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

4. If $x + 1 = 6 + 9$, which of the following is true:

   a. $x = 15$
   b. $x = 14$
   c. $x = 7$

   Why?____

   Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

5. Given that $y + 10 = 19$, is the following equation true:
   $y + 10 - 10 = 19 - 10$

   a. Yes
   b. No
   c. It depends on the value of y

   Why?____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

6. The solution to the equation $3a = 18$ is $a = 6$.
What is the solution to $3a – 2 = 18 – 2$?

    a. $a = 2$
    b. $a = 16$
    c. $a = 6$

Why?____
____
____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

7. "Sandwiches cost *s* euros and rolls cost *r* euros in a shop." Suppose I have €50, I buy 3 sandwiches and 6 rolls and it costs €39, what does $3s + 6r = 39$ mean?

    a. I bought 3 sandwiches and 6 rolls
    b. The total money I have is 39 euros
    c. The total cost of 3 sandwiches and 6 rolls is 39 euros

Why?____
____
____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

8. "In a game of rugby, a team is awarded 5 points for a try". If *t* is the number of tries, what does $5t$ stand for?

    a. 5 points
    b. 5 tries
    c. The points awarded from tries

Why?____
____
____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

Appendix B: Sample interview protocol – Question 1

Version A

"There are 20 times as many students as teachers in the school". Using s as the number of students and t as the number of teachers, which of the following equations is correct.

a. $20s = t$
b. $s = 20t$
c. $4s = 5t$

Why?____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

Version B

"There are twice as many girls as boys in a class". If b is the number of boys, consider the number of girls in terms of b. Now let g be the number of girls in the class. Which of the following equations is correct.

a. $g = 2b$
b. $b = 2g$
c. $b = g$

Why?____

Overall, how confident are you that you completed the question successfully?

Just a guess	Not very confident	Fairly confident	I'm sure I'm right

Q: Let's have another look at one of the questions you answered on *Version A on* Questionnaire A. Here is Question 1 in *Version A*. The participant would read the question again and look at the answer they gave. [Student reads the question.]

Q: So you said that $20s = t$. Do you still agree with this answer? [If the student says no, then the interviewer will ask why not and what the new answer is and why. This is to determine if the student has been influenced by *Version B* on Questionnaire B or any other factor since answering the question. If the student already knows the correct answer and can give an appropriate explanation, then it can be assumed that the student has already overcome the misconception. If the student says they agree with their original answer, they will then be presented with the *Version B* situation.]

Q: Now let's have a look at Question 1 in *Version B*. The participant would read the question again and look at the answer they gave. [Student reads the question.]. You said that $g = 2b$. Do you still agree with your answer? [If the student says no and gives the wrong answer with an explanation indicating a misconception as in *Version A,* then the situation has lost its role as an anchor. The student will then be asked why they changed their mind. If the student, however, still agrees with their former correct response, they will then be asked . . . .]

Q: Look at this Question 1 on *Version B* and this Question 1 on *Version A*. Do you see a similarity between the two questions? [If the student says no, similarities can be pointed out, but if the student says yes, as expected …]

Q: Okay, what similarities do you see? [Similarities are established and then …] What about your answers for the two questions, are you using the same reasoning to answer both?
[If they answered yes, further probing can reveal the difference. If they answered no …]

Q: So, why did you use different reasoning to answer the two questions? [There is no difference between the reasoning of the two questions. The only difference is in the ratio of students to teachers and the ratio of girls to boys. Then …]

Q: Suppose the number of teachers is t, then the number of students is $20t$ as it is 20 times the number of teachers. [The idea is to show that *Version A* while at the same time having the student maintain their anchoring response in *Version B.* If the anchor is not a brittle anchor then this can be done.]

Q: So, if s is the number of students, what would s be equal to in terms of t ? [If the student now changes their answer to 1 in *Version A* and gives an appropriate explanation as they did in *Version B,* they will then be asked how confident they are in their new answer. They will be presented with the confidence line. If they indicate at least a 2 (fairly confident), it will be concluded that they have overcome their misconception.]

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Please cite as: Tooher, H. & Johnson, P. (2020). The role of analogies and anchors in addressing students' misconceptions with algebraic equations. *Issues in Educational Research*, 30(2), 756-781. http://www.iier.org.au/iier30/tooher.pdf